On Toroidal Functions.

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1. Every spherical surface harmonic of degree n can be expressed in the form of the series

$$\Sigma_{m=0}^{m=n} A_m P_n^m(\nu) \cos(m\phi + \alpha_m),$$

where P_n^m is an associated function of the *first* kind of degree n and order m. This function satisfies the differential equation

$$\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} - \frac{m^2 u}{1-\nu^2} + n(n+1)u = 0. \qquad (1)$$

In the theory of spherical harmonics $\nu = \cos\theta$, and therefore lies between the limits 1 and -1; but in developing the theory of associated functions there is no necessity whatever to impose this limitation on the value of ν. In fact, in the theory of the potentials of ovary ellipsoids, associated functions occur in which the argument is never less than unity,* and as one of my objects is to develop the theory of toroidal functions from a point of view which brings out their connection with ordinary associated functions, the argument will always be supposed to be not less than unity. Many of the results obtained will be found to be universally true for all real values of the argument, whilst other results, when the argument lies between 1 and -1, can easily be deduced therefrom.

It is also known that if u_0 be any solution of the equation to which (1) reduces when $m = 0$, a solution of (1) is

$$u = (\nu^2 - 1)^{\frac{1}{2}m} \frac{d^m u_0}{d\nu^m}.$$

*Prof. Hicks describes these functions (and also toroidal functions) as spherical harmonics of imaginary argument. See Phil. Trans., 1881, pp. 613 and 617. In order to understand this statement it must be borne in mind that he takes the argument to be θ, where

$$\nu = \cosh\theta = \cos\iota\theta.$$

The two kinds of associated functions are therefore defined by the equations

$$\left.\begin{aligned} P_n^m &= (\nu^2-1)^{\frac{1}{2}m}\frac{d^m P_n}{d\nu^m}, \\ Q_n^m &= (\nu^2-1)^{\frac{1}{2}m}\frac{d^m Q_n}{d\nu^m}, \end{aligned}\right\} \tag{2}$$

where P_n, Q_n are the zonal harmonics of the first and second kinds respectively. Accordingly Q_n^m is an associated function of the *second* kind of degree n and order m. In ordinary spherical harmonic analysis the factor must be changed into $(1-\nu^2)^{\frac{1}{2}m}$ in the first equation in order to avoid the unnecessary* introduction of imaginary quantities. The corresponding change in the function Q_n^m need not be considered, because it never occurs in physical investigations in which ν lies between 1 and -1.

It was shown by Laplace that P_n can be expressed in the form of the definite integral

$$P_n = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}}, \tag{3}$$

and it has been shown by Heine and myself† that Q_n can be expressed in the form

$$Q_n = \int_0^\infty \frac{d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+1}}. \tag{4}$$

From these equations combined with (2) it follows that P_n^m and Q_n^m can also be expressed in the forms of definite integrals, in which case we shall have‡

$$P_n^m = \frac{(-)^m n!}{(n-m)!}\int_0^\pi \frac{\cos m\phi\, d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+1}}, \tag{5}$$

$$Q_n^m = \frac{(-)^m n!}{(n-m)!}\int_0^\infty \frac{\cosh m\phi\, d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+1}}. \tag{6}$$

The advantages of these definite integrals are that they furnish concise expressions for associated functions by means of which many of their properties may be easily investigated, and a variety of difference and mixed difference equations obtained connecting functions of different orders and degrees.

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* Those who have had occasion to employ Bessel's functions in any investigation of considerable length can hardly fail to have appreciated the superiority of the notation $I_m(x)$, $K_m(x)$' in the place of $J_m(\iota x)$, $Y_m(\iota x)$. See Hydrodynamics, Chap. XII, and Brit. Assoc. Rep., 1889, p. 28.

† Mess. Math., Vol. XIII, p. 147. ‡ Ibid., pp. 150, 152.

2. A toroidal function is an associated function of degree $n-\frac{1}{2}$ and order m; and the notation which ought in strictness to be adopted for the two kinds of toroidal functions is $P^m_{n-\frac{1}{2}}$ and $Q^m_{n-\frac{1}{2}}$; but as these functions rarely if ever occur in an investigation which also involves associated functions of integral degree n, it will be generally sufficient to employ the suffix n instead of $n-\frac{1}{2}$.

Under these circumstances we should anticipate that the expressions for the two toroidal functions in terms of definite integrals would be derived from (5) and (6) by changing n into $n-\frac{1}{2}$, so that we may write

$$P^m_n = \frac{(-)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^\pi \frac{\cos m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n+\frac{1}{2}}}, \tag{7}$$

$$Q^m_n = \frac{(-)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^\infty \frac{\cosh m\phi d\phi}{\{\nu+(\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+\frac{1}{2}}}, \tag{8}$$

and we shall presently show that this is the case.

The theory of toroidal functions was first investigated by Prof. W. M. Hicks as a means of discussing the motion of circular vortex rings. He has, however, (as frequently happens when a new branch of mathematics is being investigated for the first time), presented the subject in a somewhat complicated form. I therefore propose in the present paper to develop the subject by means of the two definite integrals (7) and (8), to correct some errors which Prof. Hicks has made, and also to extend his results.

3. Putting $n-\frac{1}{2}$ for n in (1), the differential equation for toroidal functions becomes

$$\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} - \frac{m^2u}{1-\nu^2} + (n^2-\tfrac{1}{4})u = 0, \tag{9}$$

where n is zero or any positive integer whatever, and m is zero or any positive integer which is not greater than n.

Let A^m_n denote the coefficient of the definite integral in (7); also let $D = \nu + (\nu^2-1)^{\frac{1}{2}}\cos\phi$, then from (7) we get

$$(\nu^2-1)\frac{dP^m_n}{d\nu} = -(n+\tfrac{1}{2})A^m_n\int_0^\pi \frac{\nu^2-1+\nu(\nu^2-1)^{\frac{1}{2}}\cos\phi}{D^{n+\frac{3}{2}}}\cos m\phi d\phi. \tag{10}$$

Now
$$A^m_{n+1} = \frac{n+\frac{1}{2}}{n-m+\frac{1}{2}}A^m_n,$$

whence

$$(\nu^2-1)\frac{dP^m_n}{d\nu} = -(n+\tfrac{1}{2})\left(\nu P^m_n - \frac{n-m+\frac{1}{2}}{n+\frac{1}{2}}P^m_{n+1}\right). \tag{11}$$

Again, from (10) we obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(n+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}A_n^m\left\{\int_0^\pi \frac{\cos\phi\cos m\phi d\phi}{D^{n+\frac{1}{2}}} + (\nu^2-1)^{\frac{1}{2}}\int_0^\pi \frac{\sin^2\phi\cos m\phi d\phi}{D^{n+\frac{3}{2}}}\right\}.$$

Integrating the last term by parts we obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(\nu^2-1)^{\frac{1}{2}}A_n^m\left\{(n-\tfrac{1}{2})\int_0^\pi \frac{\cos\phi\cos m\phi d\phi}{D^{n+\frac{1}{2}}} + m\int_0^\pi \frac{\sin\phi\sin m\phi d\phi}{D^{n+\frac{1}{2}}}\right\}. \qquad (12)$$

Integrating the last term of (12) by parts we finally obtain

$$(\nu^2-1)\frac{dP_n^m}{d\nu} = -(n-\tfrac{1}{2})\left(\frac{n-\frac{1}{2}}{n-m-\frac{1}{2}}P_{n-1}^m - \nu P_n^m\right) + \frac{m^2P_{n-1}^m}{n-m-\frac{1}{2}}. \qquad (13)$$

Again, from (11) we get

$$\frac{d}{d\nu}(1-\nu^2)\frac{dP_n^m}{d\nu} = (n+\tfrac{1}{2})\left(P_n^m + \nu\frac{dP_n^m}{d\nu} - \frac{n-m+\frac{1}{2}}{n+\frac{1}{2}}\frac{dP_{n+1}^m}{d\nu}\right).$$

Substituting the value of $dP_n^m/d\nu$ from (11), and that of $dP_{n+1}^m/d\nu$ from (13), the right-hand side becomes

$$-(n^2-\tfrac{1}{4})P_n^m + \frac{m^2P_n^m}{1-\nu^2},$$

which shows that the function P_n^m as defined by (7) satisfies the differential equation (9).

Eliminating $dP_n^m/d\nu$ between (11) and (13), we obtain the sequence equation

$$(n-m+\tfrac{1}{2})P_{n+1}^m - 2n\nu P_n^m + \frac{(n-\frac{1}{2})^2}{(n-m-\frac{1}{2})}P_{n-1}^m = \frac{m^2P_{n-1}^m}{n-m-\frac{1}{2}}. \qquad (14)$$

Equations (11), (13) and (14) are analogous to equations (55), (54) and (56) of §273 of my Hydrodynamics, to which they reduce when $m=0$. In fact the whole investigation is on all fours with that section, excepting that it is more general since m is not supposed to be zero; and the simplification which is obtained by first considering the case of m zero and afterwards proceeding to the more general case in which m is a positive integer, is so very slight that it is better to commence with the general case.

4. Prof. Hicks has obtained equations corresponding to (11), (13) and (14), viz. equations (26) and (27) on page 631 of his paper, which I believe are erroneous. His notation is somewhat different from mine, and I shall therefore explain it in order that the reader may be able to examine the question. He writes

$$C = \cosh u = \nu ,$$
$$S = \sinh u = (\nu^2 - 1)^{\frac{1}{2}},$$

and on page 636 he defines the toroidal function of the first kind by the equation

$$P_{m,\,n} = \int_0^\pi \frac{\cos m\theta d\theta}{(C - S\cos\theta)^{n+\frac{1}{2}}} ,$$

from which it follows that

$$P_n^m = - A_n^m P_{m,\,n} .$$

Also the accents denote differentiation with respect to u, so that

$$A_n^m P'_{m,\,n} = - S\frac{dP_n^m}{d\nu} .$$

It therefore follows that equations (26) of his paper ought to be

$$2SP'_{m,\,n} = (2n+1)(P_{m,\,n+1} - CP_{m,\,n}) ,$$
$$2SP'_{m,\,n} = (2n-1)(CP_{m,\,n} - P_{m,\,n-1}) + \frac{4m^2 P_{m,\,n-1}}{2n-1} ,$$

and consequently his sequence equation (27) is also wrong.

5. We shall now show that the expression for P_n^m satisfies (2).

If P_n^m be any function of ν which satisfies (2), it follows at once that

$$P_n^{m+1} = (\nu^2 - 1)^{\frac{1}{2}} \frac{dP_n^m}{d\nu} - \frac{m\nu}{(\nu^2 - 1)^{\frac{1}{2}}} P_n^m , \tag{15}$$

and we have to show that if the given value of P_n^m is substituted in the right-hand side, the result is equal to P_n^{m+1}.

Assuming the given value of P_n^m, the value of the first term in (15) is determined by (12). The second term gives

$$\frac{m\nu}{A_n^m(\nu^2 - 1)^{\frac{1}{2}}} P_n^m = \frac{m\nu}{(\nu^2 - 1)^{\frac{1}{2}}} \int_0^\pi \frac{\cos m\phi d\phi}{D^{n+\frac{1}{2}}} .$$

Integrating by parts, the right-hand side is equal to

$$-(n+\tfrac{1}{2})\,\nu\int_0^\pi \frac{\sin\phi\sin m\phi d\phi}{D^{n+\frac{3}{2}}}$$

$$=-(n+\tfrac{1}{2})\int_0^\pi \frac{\sin\phi\ \sin m\phi d\phi}{D^{n+\frac{1}{2}}}+(n+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}\int_0^\pi \frac{\sin\phi\ \cos\phi\ \sin m\phi d\phi}{D^{n+\frac{3}{2}}}$$

Integrating the second term on the right-hand side by parts, it becomes

$$-\int_0^\pi \frac{m\cos\phi\cos m\phi-\sin\phi\sin m\phi}{D^{n+\frac{1}{2}}}\,d\phi,$$

whence

$$\frac{m\nu}{A_n^m(\nu^2-1)^{\frac{1}{2}}}P_n^m=-(n-\tfrac{1}{2})\int_0^\pi \frac{\sin\phi\ \sin m\phi d\phi}{D^{n+\frac{1}{2}}}-m\int_0^\pi \frac{\cos\phi\ \cos m\phi d\phi}{D^{n+\frac{1}{2}}}.$$

Accordingly by (12) the right-hand side of (15) becomes

$$-A_n^m(n-m-\tfrac{1}{2})\int_0^\pi \frac{\cos(m+1)\phi}{D^{n+\frac{1}{2}}}\,d\phi.$$

From the value of A_n^m it follows that

$$-A_n^m(n-m-\tfrac{1}{2})=A_n^{m+1},$$

and therefore the right-hand side of (15) is equal to P_n^{m+1}, which shows that the expression (7) satisfies (2). We have therefore shown that the definite integral (7) satisfies all the necessary conditions.

6. Equations (11) and (13) furnish two equations which connect the differential coefficient of a function of order m and degree n with the functions of order m and degrees $n+1$, n and $n-1$, from which results the sequence equation (14) connecting three functions of order m and degrees $n+1$, n and $n-1$. We shall now establish three similar equations connecting functions of degree n and orders $m+1$, m and $m-1$. The first equation of the latter class is (15); to obtain a second equation we observe that (12) may be written in the form

$$(\nu^2-1)^{\frac{1}{2}}\frac{dP_n^m}{d\nu}=-A_n^m\left\{\frac{n-m-\frac{1}{2}}{2A_n^{m+1}}P_n^{m+1}+\frac{n+m-\frac{1}{2}}{2A_n^{m-1}}P_n^{m-1}\right\}$$

$$=\tfrac{1}{2}P_n^{m+1}+\tfrac{1}{2}(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})P_n^{m-1}. \quad (16)$$

Eliminating $dP_n^m/d\nu$ between (15) and (16), we get

$$P_n^{m+1}+\frac{2m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m=(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})P_n^{m-1}. \quad (17)$$

By means of (17), equation (15) may be written

$$(\nu^2-1)\frac{dP_n^m}{d\nu}=(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})(\nu^2-1)^{\frac{1}{2}}P_n^{m-1}-m\nu P_n^m, \tag{18}$$

whilst (15) is

$$(\nu^2-1)\frac{dP_n^m}{d\nu}=(\nu^2-1)^{\frac{1}{2}}P_n^{m+1}+m\nu P_n^m. \tag{19}$$

7. Equations (17), (18) and (19) connect three toroidal functions of degree n and orders $m+1$, m and $m-1$, and they are the analogues of (14), (13) and (11). The corresponding equations given by Prof. Hicks are (32) and (31) on p. 633 of his paper, and we shall now show that his equations are wrong.

From (18), the equation corresponding to the second of his equations (31) ought to be

$$2SP'_{m,n}=-2mCP_{m,n}-(2n+2m-1)SP_{m-1,n},$$

whilst that corresponding to his first equation is

$$2SP'_{m,n}=2mCP_{m,n}-(2n-2m-1)SP_{m+1,n},$$

which shows that his sequence equation (32) is also wrong.

8. In the theory of toroidal functions, n is any positive integer, and m is any positive integer which is not greater than n, the value zero being included. Now, if in §§ 3 to 6 we write $n+\frac{1}{2}$ for n, the whole of the preceding analysis will apply to associated functions of the first kind of degree n and order m. The definite integral expression (7) becomes equal to (5), and the three sequence equations (11), (13) and (14), connecting associated functions of order m and degrees $n+1$, n and $n-1$, become

$$\left.\begin{aligned}
&(\nu^2-1)\frac{dP_n^m}{d\nu}=-(n+1)\left(\nu P_n^m-\frac{n-m+1}{n+1}P_{n+1}^m\right),\\
&(\nu^2-1)\frac{dP_n^m}{d\nu}=-n\left(\frac{n}{n-m}P_{n-1}^m-\nu P_n^m\right)+\frac{m^2P_{n-1}^m}{n-m},\\
&(n-m+1)P_{n+1}^m-(2n+1)\nu P_n^m+\frac{n^2}{n-m}P_{n-1}^m=\frac{m^2P_{n-1}^m}{n-m},
\end{aligned}\right\} \tag{20}$$

whilst the three equations (19), (18) and (17) connecting the associated functions degree n and orders $m+1$, m and $m-1$, become

$$\left.\begin{aligned} (\nu^2-1)\frac{dP_n^m}{d\nu} &= (\nu^2-1)\,P_n^{m+1} + m\nu P_n^m, \\ (\nu^2-1)\frac{dP_n^m}{d\nu} &= (n+m)(n-m+1)(\nu^2-1)^{\frac{1}{2}}P_n^{m-1} - m\nu P_n^m, \\ P_n^{m+1} + \frac{2m\nu}{(\nu^2-1)^{\frac{1}{2}}}P_n^m &= (n+m)(n-m+1)\,P_n^{m-1}. \end{aligned}\right\} \quad (21)$$

These formulæ can be proved directly from equation (5).

9. The definite integral expression (8) for toroidal functions of the second kind can be shown by a precisely similar method to satisfy the same equations, but in deducing equations such as (13) or (14) it must be recollected that m must be supposed to be not greater than $n-1$. Similar observations apply to the ordinary associated functions of the second kind, which can be deduced by writing $n+\frac{1}{2}$ for n.

There is one result which is of considerable importance in physical investigations, viz. the value of the quantity

$$P_n'^m Q_n^m - Q_n'^m P_n^m,$$

where the accents denote differentiation with respect to ν. Calling this u_n^m, substitute the values of $P_n'^m$ and $Q_n'^m$ from (18) and we get

$$u_n^m = \frac{(n+m-\frac{1}{2})(n-m+\frac{1}{2})}{(\nu^2-1)^{\frac{1}{2}}}\,(P_n^{m-1}Q_n^m - P_n^m Q_n^{m-1}).$$

In (19) write $m-1$ for m, and then substitute in the last equation the values of Q_n^m and P_n^m and we get

$$u_n^m = -(n+m-\tfrac{1}{2})(n-m+\tfrac{1}{2})\,u_n^{m-1};$$

accordingly,

$$u_n^m = (-)^m(n+\tfrac{1}{2})(n+\tfrac{3}{2})\ldots\ldots(n+m-\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots\ldots(n-m+\tfrac{1}{2})\,u_n.$$

The value of u_n is $\pi(\nu^2-1)^{-\frac{1}{2}}$ (see Hydrodynamics, §278, equations (63)).

The corresponding result for ordinary associated functions is

$$u_n^m = (-)^m (n+1)(n+2)\ldots\ldots(n+m)\,n\,(n-1)\ldots\ldots(n-m+1)\,u_n.$$

Here $u_n = (\nu^2 - 1)^{-1}$, so that

$$P_n'^m Q_n^m - P_n^m Q_n'^m = \frac{(-)^m (n+m)!}{(n-m)!\,(\nu^2-1)}.$$

10. Almost, if not all, the foregoing results are given in Prof. Hicks' paper, although some of them are exhibited in an erroneous form. But in physical investigations relating to the potentials of anchor rings, $\nu = \cosh\eta$, where the equation $\eta =$ const. represents a family of such surfaces, and in these investigations η is usually a very large quantity, and consequently $\varepsilon^{-\eta}$ is very small. Accordingly, if we put $\varepsilon^{-\eta} = k$, we can expand these functions in series of powers of k, and these series furnish expressions which are of great convenience in discussing the motion of circular vortices. We shall therefore proceed to investigate the appropriate series for the two kinds of zonal toroidal functions.

Putting $m = 0$ in (8) we obtain

$$Q_n = \int_0^\infty \frac{d\phi}{\{\nu + (\nu^2-1)^{\frac{1}{2}}\cosh\phi\}^{n+\frac{1}{2}}}.$$

In this write

$$\nu = \tfrac{1}{2}(k + k^{-1}),$$
$$pk = \{\nu + (\nu^2 - 1)^{\frac{1}{2}}\cosh\phi\}^{-1},$$

and the integral becomes

$$Q_n = k^{n+\frac{1}{2}} \int_0^1 \frac{p^{n-\frac{1}{2}}dp}{(1-p)^{\frac{1}{2}}(1-k^2p)^{\frac{1}{2}}}$$

$$= 2k^{n+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\theta\, d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}}, \qquad (22)$$

if $p = \sin^2\theta$.

From (22) it follows that

$$\left.\begin{aligned} Q_0 &= 2k^{\frac{1}{2}}F(k), \\ Q_1 &= 2k^{-\frac{1}{2}}(F - E), \end{aligned}\right\} \qquad (23)$$

where F and E are the first and second elliptic integrals to modulus k.

Let H_s denote the coefficient of x^s in the expansion for $(1-x)^{-\frac{1}{2}}$, so that when s is not zero,

$$H_s = \frac{1.3.5\ldots.(2s-1)}{2.4.6\ldots.2s}$$

whilst $H_0 = 1$; then since

$$\int_0^{\frac{1}{2}\pi} \sin^{2s}\theta d\theta = \tfrac{1}{2}\pi H_s,$$

we obtain
$$Q_n = \pi k^{n+\frac{1}{2}} \Sigma_{s=0}^{s=\infty} H_s H_{n+s} k^{2s}. \tag{24}$$

This is the series for Q_n in powers of k.

11. The function P_n is more difficult to deal with. From (7) we have

$$P_n = \int_0^\pi \frac{d\phi}{\{\nu + (\nu^2+1)^{\frac{1}{2}}\cos\phi\}^{n+\frac{1}{2}}},$$

and by means of a well-known transformation, this can be expressed in the form

$$P_n = \int_0^\pi \{\nu + (\nu^2-1)^{\frac{1}{2}}\cos\phi\}^{n-\frac{1}{2}} d\phi.$$

Putting $\nu = \frac{1}{2}(k+k^{-1})$, this becomes

$$P_n = 2k^{-n+\frac{1}{2}} \int_0^\pi (1 - k'^2 \sin^2\theta)^{n-\frac{1}{2}} d\theta, \tag{25}$$

from which it follows that

$$\left.\begin{aligned} P_0 &= 2k^{\frac{1}{2}} F(k'), \\ P_1 &= 2k^{-\frac{1}{2}} E(k'). \end{aligned}\right\} \tag{26}$$

From equation (25) it follows that P_n is a species of generalized elliptic integral; and since the elliptic integrals of the first and second kinds, when k' is nearly equal to unity, are known to be capable of expansion in a series of the form

$$u \log 4/k + v,$$

where u and v are series proceeding according to even powers of k^2, we should

anticipate that the definite integral in (25) is capable of being expressed in a similar manner.* This we shall now show to be the case.

In (9) put $m = 0$, $\nu = \frac{1}{2}(k + k^{-1})$, and it becomes

$$k^2 \frac{d^2u}{dk^2} - \frac{2k^3}{1-k^2} \frac{du}{dk} - (n^2 - \tfrac{1}{4}) u = 0$$

In this write $u = \eta k^{-n+\frac{1}{2}}$, and the equation for η is

$$(1-k^2) \frac{d^2\eta}{dk^2} - \{(2n-1) - (2n-3) k^2\} k^{-1} \frac{d\eta}{dk} + (2n-1)\eta = 0. \quad (27)$$

Assume
$$\eta = \phi_n(k) \log 4/k + \psi_n(k). \quad (28)$$

Substituting in (27), it will be found that the equation will be satisfied, provided ϕ_n satisfies an equation of the same form as (27), whilst ψ_n satisfies the equation

$$(1-k^2) \frac{d^2\psi}{dk^2} - \{2n - 1 - (2n-3) k^2\} k^{-1} \frac{d\psi}{dk}$$
$$+ (2n-1)\psi - 2(1-k^2)k^{-1} \frac{d\phi}{dk} + \{2n - (2n-2)k^2\} \frac{\phi}{k^2} = 0. \quad (29)$$

To find the value of ϕ_n, assume

$$\phi_n = \Sigma A_{2s} k^{2s},$$

and substitute in (27) and we get

$$2s(2s - 2n) A_{2s} k^{2s-2}$$
$$- \{(2s+1)(2s - 2n + 1) A_{2s} - (2s+2)(2s - 2n + 2) A_{2s+2}\} k^{2s} - \ldots = 0.$$

This equation will be satisfied provided the first term begins with $s = 0$ or $s = n$; but inasmuch as it is known that in the expression for $E(k')$, the first term of

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* The neatest way that I have met with of proving the series for $F(k')$, when k' is nearly equal to unity, is contained in a paper by Prof. Sylvester, Phil. Mag., XX (1860), p. 528. His method leads to the remarkable result that

$$\int_0^{\frac{1}{2}\pi} \frac{\log \cos \phi d\phi}{(1 - k^2 \cos^2 \phi)^{\frac{1}{2}}} = -\int_0^{\frac{1}{2}\pi} \frac{\log \{1 + (1 - k^2 \cos^2 \phi)^{\frac{1}{2}}\} d\phi}{(1 - k^2 \cos^2 \phi)^{\frac{1}{2}}}$$

the series by which $\log 4/k$ is multiplied is $\frac{1}{2}k^2$, it follows that we must take $s=n$, so that the first term in the series for ϕ_n is k^{2n}. Also

$$A_{2s+2}=\frac{(2s+1)(2s-2n+1)}{(2s+2)(2s-2n+2)}A_{2s},$$

whence

$$A_{2n+2s}=\frac{(2n+1)(2n+3)\ldots\ldots(2n+2s-1)}{(2n+2)(2n+4)\ldots\ldots(2n+2s)}H_s A_{2n}$$

and

$$\phi_n=A_{2n}k^{2n}\left\{1+\frac{2n+1}{2.\,2n+2}k^2+\frac{1.3.(2n+1)(2n+3)}{2.4.(2n+2)(2n+4)}k^4+\ldots\ldots\right\} \tag{30}$$

We shall now show that $A_{2n}=2H_n$. Since ϕ_n satisfies the differential equation (27), it follows that $\phi_n k^{-n+\frac{1}{2}}$ satisfies the differential equation for zonal toroidal functions, and accordingly if A_{2n} be suitably chosen, the series (30) will satisfy the sequence equation (14). Putting $m=0$ and $\nu=\frac{1}{2}(k+k^{-1})$, this equation becomes

$$(n+\tfrac{1}{2})P_{n+1}-n(1+k^2)k^{-1}P_n+(n-\tfrac{1}{2})P_{n-1}=0. \tag{31}$$

On substitution, this will be found to be satisfied provided $A_{2n}=2H_n$; also if we put $n=0$ and $n=1$, it will be found that the series for ϕ_0 and ϕ_1 are respectively the coefficients of $\log 4/k$ in the series for $2F(k')$ and $2E(k')$. We therefore finally obtain for the portion of P_n which involves $\log 4/k$ the expression

$$2k^{n+\frac{1}{2}}\log 4/k\,\Sigma_{s=0}^{s=\infty}H_sH_{n+s}k^{2s}=(2/\pi)\,Q_n\log 4/k \tag{32}$$

by (24), whilst

$$\phi_n=2k^{2n}\Sigma_{s=0}^{s=\infty}H_sH_{n+s}k^{2s}. \tag{33}$$

12. To obtain the series for ψ_n we must revert to the differential equation (29), and we must first arrange the last two terms in a series of powers of k. The series for ϕ_n may be written in the form

$$\phi_n=2H_nk^{2n}\Sigma_1^\infty(1+A_{2s}k^{2s}),$$

where

$$A_{2s}=H_s\frac{(2n+1)(2n+3)\ldots\ldots(2n+2s-1)}{(2n+2)(2n+4)\ldots\ldots(2n+2s)} \tag{34}$$

Substituting in the last two terms of (29), and rearranging according to powers of k, the differential equation becomes

$$(1-k^2)\frac{d^2\psi}{dk^2}-\{2n-1-(2n-3)k^2\}\frac{1}{k}\frac{d\psi}{dk}+(2n-1)\psi$$

$$+2H_n\left[-2k^{2n-2}+\Sigma_0^\infty\left\{\frac{2n+2s+1}{2s+2}+\frac{2s+1}{2n+2s+2}\right\}A_{2s}k^{2n+2s}\right]=0. \quad (35)$$

The case of $n=0$ would merely give us the series for $F(k')$ which is considered in all books on Elliptic Functions. When n is equal to unity, it is known from the series for $E(k')$ that the series for ψ begins with a term which is independent of k. We shall therefore assume that

$$\psi=\Sigma_0^\infty B_{2r}k^{2r}$$

Substituting in (35) we get

$$\Sigma_0^\infty\{(2r+1)(2n-2r-1)B_{2r}-(2r+2)(2n-2r-2)B_{2r+2}\}k^{2r}$$

$$+2H_n\left[-2nk^{2n-2}+\Sigma_0^\infty\left\{\frac{2n+2s+1}{2s+2}+\frac{2s+1}{2n+2s+2}\right\}A_{2s}k^{2n+2s}\right]=0. \quad (36)$$

If, therefore, r is not greater than $n-1$, which can only happen when n is not less than unity, we get

$$B_{2r+2}=\frac{(2r+1)(2n-2r-1)}{(2r+2)(2n-2r-2)}B_{2r}. \quad (37)$$

We therefore see that ψ consists of two parts, viz. a terminating series the law of whose coefficients is independent of ϕ, and whose values are given by (37), and an infinite series which we shall presently investigate. Putting $r=n-1$ in (36), we see that

$$B_{2n-2}=2H_{n-1},$$

from which it follows that the terminating series is

$$\frac{2}{H_{n-1}}\left\{1+\frac{2n-1}{2(2n-2)}k^2+\frac{1.3.(2n-1)(2n-3)}{2.4.(2n-2)(2n-4)}k^4+\dots\right.$$

$$\left.+H_{n-2}\frac{(2n-1)(2n-3)\dots 7.5}{(2n-2)(2n-4)\dots 6.4}k^{2n-4}+H_{n-1}^2k^{2n-2}\right\}. \quad (38)$$

To find the infinite series, put $r = n$ in (36), and recollecting the value of A_{2s}, we get

$$B_{2n+2} = \frac{2n+1}{2.2n+2} B_{2n} - H_{n+1} \left\{ \frac{1}{1.2} + \frac{1}{(2n+1)(2n+2)} \right\}, \tag{39}$$

and generally

$$B_{2n+2s+2} = \frac{(2s+1)(2n+2s+1)}{(2s+2)(2n+2s+2)} B_{2n+2s}$$
$$- 2H_{s+1}H_{n+s+1} \left\{ \frac{1}{(2s+1)(2s+2)} + \frac{1}{(2n+2s+1)(2n+2s+2)} \right\}. \tag{40}$$

13. Equation (40) enables us to obtain all the coefficients in terms of B_{2n} which is undetermined, and we shall now proceed to find its value.

The portion of P_n which involves $\log 4/k$ is $\phi_n k^{-n+\frac{1}{2}} \log 4/k$; and since by (32) and (33) this is equal to $(2/\pi)\, Q_n \log 4/k$, it follows that $\phi_n k^{-n+\frac{1}{2}}$ satisfies the sequence equation (31), as can be readily verified by actual substitution; whence $\psi_n k^{-n+\frac{1}{2}}$ also satisfies (31), and consequently ψ_n satisfies the equation

$$(2n+1)\,\psi_{n+1} - 2n\,(1+k^2)\,\psi_n + (2n-1)\,k^2\psi_{n-1} = 0. \tag{41}$$

In this write $\psi_n = \Sigma_{r=0}^{r=\infty} B_{2r}^n k^{2r}$,

where the index n denotes the *degree* of the function to which B belongs, whilst the suffix $2r$ denotes the *power* of k of which it is the coefficient. The quantity which we require to determine is the coefficient of k^{2n} in the series for ψ_n, and in this notation it is represented by B_{2n}^n.

Substituting the above value of ψ_n in (41), and equating coefficients of k^{2r}, we get

$$(2n+1)\,B_{2r}^{n+1} - 2n\,(B_{2r}^n + B_{2r-2}^n) + (2n-1)\,B_{2r-2}^{n-1} = 0. \tag{42}$$

We have already shown that $B_{2n-2}^n = 2H_{n-1}$, whence, writing $r = n$ in (42),

we get
$$B_{2n}^n = \frac{2n-1}{2n} B_{2n-2}^{n-1} - \frac{2H_n}{2n\,(2n-1)}. \tag{43}$$

Now
$$P_1 = 2k^{-\frac{1}{2}} E(k') = k^{-\frac{1}{2}} (\phi_1 \log 4/k + \psi_1),$$

and consequently B_2^1, which is the coefficient of k^2 in the expansion of ψ_1, is

equal to twice the coefficient of k^2 in that portion of the series for $E(k')$ which does not involve $\log 4/k$; accordingly

$$B_2^1 = -\tfrac{1}{2},$$

whence by (43) we obtain

$$\begin{aligned} B_{2n}^n &= -2H_n\left\{\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots\frac{1}{(2n-1)\,2n}\right\} \qquad (44)\\ &= -2H_nS_n, \end{aligned}$$

where S_n denotes the above series. Going back to (40) and using this value of B_{2n}^n, we see that we may write

$$B_{2n+2s}^n = -2H_sH_{n+s}(S_s+S_{n+s}),$$

in which the symbols H_0 and S_0 must be interpreted as being respectively equal to unity and zero.

The second part of the series for ψ_n is therefore equal to

$$-2\Sigma_{s=0}^{s=\infty}H_sH_{n+s}(S_s+S_{n+s})\,k^{2n+2s},$$

and the value of P_n finally becomes

$$\begin{aligned} P_n = {} & 2k^{n+\frac{1}{2}}\log 4/k\,\Sigma_{s=0}^{s=\infty}H_sH_{n+s}k^{2s}\\ & +\frac{2}{H_{n-1}}\left\{1+\frac{2n-1}{2.\,2n-2}k^2+\frac{1.3.2n-1.2n-3}{2.4.2n-2.2n-4}k^4+\ldots\right.\\ & \left.+H_{n-2}\frac{2n-1.2n-3\ldots.7.5}{2n-2.2n-4\ldots.6.4}k^{2n-4}+H_{n-1}^2k^{2n-2}\right\}k^{-n+\frac{1}{2}}\\ & -2k^{n+\frac{1}{2}}\Sigma_{s=0}^{s=\infty}H_sH_{n+s}(S_s+S_{n+s})\,k^{2s}. \qquad (45) \end{aligned}$$

This series holds good for all positive integral values of n, *excluding* zero. When $n=0$, the value of P_0 may be deduced from the known series for $F(k')$, and when $n=1$, the series (45) reduces to the known series for $2k^{-\frac{1}{2}}E(k')$.

The first few terms of the series for the first five zonal functions are given by Prof. Hicks,* but as he has obtained them by direct calculation from the sequence equation (31) combined with the series for $F(k')$ and $E(k')$, his results do not show the law of formation of the coefficients.

14. It is quite obvious that the definite integrals (7) and (8), which give the values of the functions of any order and degree, could be expressed by

* Phil. Trans., 1884, pp. 171, 172.

series analogous to (45) and (24), but the results would be of little physical interest, as the utility of toroidal functions consists in their applications to the motion of circular vortex rings. The motion in most cases of practical interest can be obtained by means of Stokes' current function; and this function, as is well known, can be made to depend on a solution of Laplace's equation which involves harmonics of degree n and order unity. We therefore require to investigate the functions P_n^1, Q_n^1. Putting $m = 0$ in (11) we obtain

$$(\nu^2 - 1)^{\frac{1}{2}} P_n^1 = (n + \tfrac{1}{2})(P_{n+1} - \nu P_n), \tag{46}$$

with a similar equation for Q. Now the functions which are most useful in investigating the motion of circular vortex rings are two functions R and T (which are different from Prof. Hicks' R and T), and satisfy the equations*

$$(\nu^2 - 1)^{\frac{1}{2}} P_n^1 = k^{-n-\frac{1}{2}} R_n,$$
$$(\nu^2 - 1)^{\frac{1}{2}} Q_n^1 = -\tfrac{1}{4}\pi k^{n-\frac{1}{2}} T_n,$$

so that from (46) we get

$$R_n = \tfrac{1}{4}(2n + 1)\{2kP_{n+1} - (1 + k^2) P_n\} k^{n-\frac{1}{2}},$$
$$T_n = -\pi^{-1}(2n + 1)\{2kQ_{n+1} - (1 + k^2) Q_n\} k^{-n-\frac{1}{2}}.$$

Since only a few terms of the series for R and T are required in physical investigations, they may be easily calculated from the above formulæ by means of (45) and (24).

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* See my Treatise on Hydrodynamics, Chap. XII, equations (65) and (71).